Measuring Up to Stiglitz-Weiss: Measure Theory, Credit and Information Dispersion

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Abstract

This paper generalizes the classic Stiglitz-Weiss model using measure theory and integration to better analyze creditworthiness under information asymmetries. We revisit the original model, address its limitations, and introduce a measure-theoretic framework to handle the distribution of borrower types and associated risks. Our model accounts for a broad spectrum of borrower behaviors, including those underrepresented in traditional models, and derives new equilibrium conditions and credit rationing outcomes. We introduce a novel information asymmetry, termed information dispersion, where gaining information about one dimension of a borrower's type increases uncertainty about other dimensions. This arises from the multidimensional nature of borrower types and the dynamic information structure in informal markets. Simulations illustrate this concept. Our findings offer a rigorous approach to understanding credit markets under information asymmetries, potentially improving credit access and financial stability.

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1 Introduction

> COWEN: Now, your best-cited piece is your 1981 article with Andy Weiss on credit rationing, which is a macroeconomic idea. But do you think that since then, the real problem has more often been that we've thrown too much credit at things? So, the housing bubble, the student loan crisis — wouldn't we have been better off with a lot more credit rationing?
>
> STIGLITZ: The issue here was that we weren't very good at credit allocation and that we thought, let the market rip. We lowered interest rates. We deregulated, so we didn't look at where the credit was going. The bank supervisors the Federal Reserve is supposed to oversee — and there are actually several other supervisors that are supposed to oversee the riskiness of the lending — that's where the fault came.

–Joseph Stiglitz interview with Tyler Cowen, *Conversations with Tyler,* June 26, 2024
https://conversationswithtyler.com/episodes/joseph-stiglitz/

As the quote highlights, the implications of the Stiglitz-Weiss model are still being debated. This paper aims to provide an even more nuanced mathematical framework for understanding these issues. By now, the seminal work of Stiglitz and Weiss (1981) on credit rationing under information asymmetries has been a cornerstone in our understanding of credit markets for over four decades. Their model elegantly demonstrated how adverse selection and moral hazard could lead to equilibrium credit rationing, even in the absence of price rigidities. While this framework has proven invaluable in explaining various phenomena in credit markets, the increasing complexity of modern financial systems and the heterogeneity of borrower behaviors in necessitate a more nuanced approach.

This paper proposes a generalization of the Stiglitz and Weiss model by incorporating advanced techniques from measure theory and integration. Our approach provides a more comprehensive framework for analyzing creditworthiness in the presence of information asymmetries.

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space representing the set of all possible borrower types, where Ω is the sample space, $\mathcal{F}$ is a σ-algebra on Ω, and μ is a probability measure. We define a risk function

$r : \Omega \to \mathbb{R}_+$ that maps each borrower type to its associated risk level. The expected risk for a given subset of borrowers $A \in \mathcal{F}$ can then be expressed as:

$$\mathbb{E}[r(A)] = \int_A r(\omega) d\mu(\omega)$$

This formulation allows us to capture the full spectrum of borrower behaviors and risk profiles, including those that are typically underrepresented in traditional models.

Building on this foundation, we develop a series of theorems that extend the Stiglitz and Weiss framework:

Theorem 1. Under the generalized measure-theoretic model, there exists a unique equilibrium interest rate r^* that maximizes the lender's expected return, given by:

$$r^* = \arg\max_{r \in \mathbb{R}_+} \int_\Omega \pi(r, \omega) d\mu(\omega)$$

where $\pi(r, \omega)$ represents the lender's profit function for a given interest rate r and borrower type ω.

Theorem 2. Credit rationing occurs in equilibrium if and only if:

$$\frac{\partial}{\partial r} \int_\Omega \pi(r^*, \omega) d\mu(\omega) = 0$$

and

$$\int_\Omega D(r^*, \omega) d\mu(\omega) > S(r^*)$$

where $D(r, \omega)$ is the demand function for credit by borrower type ω at interest rate r, and $S(r)$ is the supply function of loanable funds.

These theorems provide a rigorous mathematical foundation for analyzing credit rationing in a more general setting, accounting for the continuous nature of borrower risk profiles.

We make the following contributions. First, we provide generalized moral hazard and adverse selection versions. Most importantly, however, we introduce a new information asymmetry that arises organically from the generalized setting. I call this **information dispersion**. Information

Dispersion occurs when gaining information about one dimension of a borrower's type increases uncertainty about other dimensions. This new problem arises from the multidimensional nature of borrower types and the dynamic information structure in informal markets. I discuss this phenomenon at length in Section 4.

Our paper proceeds as follows. We first provide a detailed review of the Stiglitz and Weiss model, highlighting its key assumptions and limitations. We then introduce the measure-theoretic framework and develop the generalized model. We next present our main theoretical results, including proofs of Theorems 1 and 2, as well as additional corollaries that arise from our framework. This is where we introduce information dispersion and illustrate the concept with simulations. Afterwards we discuss the implications of our findings, particularly in the context of improving credit access and financial stability. We finally conclude.

By enhancing the mathematical sophistication of credit rationing models, this paper aims to bridge the gap between theoretical economics and the complex realities of creditworthiness-oriented markets in modern finance such as private equity and other contexts where information asymmetries may be stark. Our approach not only generalizes existing results but also provides a flexible framework for incorporating additional factors that influence creditworthiness decisions, paving the way for more accurate and nuanced analyses of credit allocation in the presence of information asymmetries.

2 Literature Review

The literature on credit rationing and information asymmetries has evolved significantly since the work of Stiglitz and Weiss[1].Excellent reviews of related modern finance work are in Amiram et al (2017) and Gambacorta et al (2023).

Adverse selection and moral hazard are two critical issues in credit markets. Adverse selection occurs when lenders cannot distinguish between high-risk and low-risk borrowers, leading to a pool of borrowers that is riskier on average. Moral hazard arises when borrowers engage in riskier behavior after obtaining a loan, knowing that the lender bears some of the risk. The Stiglitz-Weiss model

[1] Other classic readings are Bester (1985) and Jaffee and Russell, (1976).

elegantly captures these phenomena, and subsequent research has expanded on their framework to explore various market conditions and borrower behaviors.

Several studies have extended the Stiglitz-Weiss model to address its limitations and apply it to different contexts before us. For instance, Besanko and Thakor (1987) introduced collateral requirements to mitigate adverse selection, while Williamson (1987) incorporated costly state verification to address moral hazard. Other work has focused on the role of relationship lending and dynamic interactions between borrowers and lenders (Boot, 2000; Petersen and Rajan, 1994). While information has traditionally been viewed as a means to an end, recent research suggests that information itself can be a component of the utility function (see Golman, Hagmann and Loewenstein (2017) for a review).

Information asymmetries can be significant in many modern environments, such as private equity, where social contexts may play a role (e.g. Johan and Zhang, 2021). Firm insiders, may have better information than do market participants on the value of their firm's assets and investment opportunities. Private equity firms often use significant debt to finance their acquisitions. This makes the creditworthiness of the acquired company crucial. A company with a strong balance sheet, stable cash flow, and a solid industry position is more likely to secure favorable debt terms. The creditworthiness of a private equity firm itself can indirectly influence investor confidence. A firm with a history of successful exits and a strong balance sheet is more likely to attract investors for future funds. The success of an exit (e.g., IPO, sale to another company) often depends on the company's financial health, which is closely linked to its creditworthiness. An extension of the classic framework is justified due to the opaque nature of many private firms in this context.

Empirical studies and simulations have played a crucial role in validating theoretical models of credit markets and have been used to test hypotheses related to information asymmetries and credit rationing (Karlan and Zinman, 2009; Giné and Klonner, 2005). The information imperfections literature is vast[2]. Relationship building and finance are studied by Berger and Udell (1995), Peterson and Rajan (1994), Boot (2008), and our framework is a general fit for their arguments as well.

The application of measure theory and integration in economics has provided powerful tools for both theory and applications. For the former, see Aliprantis and Border (2006), and for the latter,

[2]More recent works include Chodorow-Reich, et al (2022), Berg (2018), Berg et al (2016, 2021), Banerjee and Duflo, (2010, 2014); Karlan and Zinman, (2009, 2010), Banerjee, et al, (2024), de Janvry et al., (2010)

observe Kirman (1981).

Our work differs in that it specifically applies measure theory and integration to generalize the Stiglitz-Weiss model of credit rationing. The primary focus is on addressing information asymmetries in credit markets, particularly in informal markets. We build on the specific economic model of Stiglitz and Weiss, extending it using measure theory to handle the distribution of borrower types and associated risks. The paper introduces new equilibrium conditions and credit rationing outcomes, as well as the concept of information dispersion, which is a novel form of information asymmetry.

In summary, our paper applies such mathematical tools to a specific economic problem, extending the Stiglitz-Weiss model and introducing new theoretical and practical insights into credit markets. The key innovation in the paper is the introduction of information dispersion, a new problem specific to multidimensional type spaces and informal markets. This concept highlights the complexity of information acquisition and its impact on lending decisions, offering new insights into credit market dynamics. The concept of information dispersion, introduced in this paper, represents a novel form of information asymmetry that arises from the multidimensional nature of borrower types. While traditional models focus on adverse selection and moral hazard, information dispersion highlights the complexity of information acquisition in informal markets. This phenomenon occurs when gaining information about one dimension of a borrower's type increases uncertainty about other dimensions, complicating the lender's decision-making process.

3 A Stiglitz-Weiss Review

3.1 Framework and Assumptions

The Stiglitz and Weiss (1981) model of credit rationing under asymmetric information has been pivotal in understanding the dynamics of credit markets. This section provides a detailed review of their model, emphasizing its key assumptions and results.

Consider a credit market with the following characteristics:

1. A large number of borrowers, each with a potential investment project.
2. A large number of risk-neutral banks competing for deposits and loans.
3. Asymmetric information: borrowers know the risk characteristics of their projects, but banks

do not.

Let R denote the gross interest rate (1 + interest rate) charged by banks. Each borrower's project is characterized by a random return $\tilde{Y}$ with cumulative distribution function $F(Y, \theta)$, where θ represents the project's risk parameter. Higher values of θ correspond to riskier projects in the sense of mean-preserving spreads.

Assumption 1 (Limited Liability). Borrowers are protected by limited liability. If the project return is less than the amount owed, the borrower declares bankruptcy and the bank receives the entire project return.

Assumption 2 (Identical Expected Returns). All projects have the same expected return $\bar{Y}$, regardless of their risk:

$$\bar{Y} = \int_0^\infty Y dF(Y, \theta) \quad \forall \theta$$

Assumption 3 (Reservation Wage). Each borrower has a reservation wage $\bar{W}$. They will only undertake the project if their expected return exceeds $\bar{W}$.

3.2 Key Results

3.2.1 Borrower Behavior

Given these assumptions, a borrower's expected return $\pi_B(R, \theta)$ is:

$$\pi_B(R, \theta) = \int_R^\infty (Y - R) dF(Y, \theta)$$

Stiglitz and Weiss proved the following crucial result:

Theorem 2.1 (Stiglitz-Weiss). $\frac{\partial \pi_B(R, \theta)}{\partial \theta} > 0$ for all $R > 0$.

This theorem implies that riskier borrowers have a higher expected return from their projects at any given interest rate. Consequently, as the interest rate increases, less risky borrowers drop out of the market first, leading to adverse selection.

3.2.2 Bank Behavior

Banks, being risk-neutral, aim to maximize their expected return $\pi_L(R, \hat{\theta})$, where $\hat{\theta}$ is the average risk of the pool of borrowers:

$$\pi_L(R, \hat{\theta}) = R \cdot P(Y \geq R|\hat{\theta}) + E(Y|Y < R, \hat{\theta}) \cdot P(Y < R|\hat{\theta}) - \rho$$

Here, ρ represents the bank's cost of funds.

Stiglitz and Weiss demonstrated that there exists an interest rate R^* that maximizes the bank's expected return. Importantly, this rate may not clear the market:

Theorem 2.2 (Credit Rationing). There exists an equilibrium interest rate R^* such that:

1. $\pi_L(R^*, \hat{\theta}) \geq \pi_L(R, \hat{\theta})$ for all $R \neq R^*$ 2. At R^*, the demand for loans may exceed the supply

This result implies that banks may not increase the interest rate even in the presence of excess demand, as doing so would decrease their expected return due to adverse selection and increased default risk.

3.3 Limitations of the Stiglitz-Weiss Model

While groundbreaking, the Stiglitz-Weiss model has several limitations:

1. Discrete Risk Types: The model typically assumes a finite number of borrower types, which may not fully capture the continuous nature of risk in real-world settings.

2. Homogeneous Project Size: All projects are assumed to require the same loan size, which is often not the case in practice.

3. Static Framework: The model does not account for dynamic considerations such as relationship lending or reputation building.

4. Limited Risk Measures: The model primarily focuses on default risk, potentially overlooking other dimensions of credit risk.

5. Simplistic Borrower Behavior: The model assumes that borrowers make decisions based solely on expected returns, ignoring factors such as risk aversion or time preferences.

6. Uniform Distribution of Types: The distribution of borrower types is often assumed to be uniform, which may not reflect the true distribution in many credit markets.

To help drive home these points, I shall now connect each limitation to challenges in informal markets and provide theoretical representations where appropriate. The purpose of this elaboration is threefold. It:

1. Demonstrates how each limitation of the Stiglitz-Weiss model is particularly problematic when applied to modern finance.
2. Provides mathematical formulations that illustrate how these limitations could be addressed in a more comprehensive model.
3. Sets the stage for your measure-theoretic approach by highlighting the need for a more flexible and nuanced framework.

3.4 Limitations of the Stiglitz-Weiss Model and Informal Market Challenges

We unpack the limitations in detail now. While the Stiglitz-Weiss model provides crucial insights into credit rationing, it has several limitations, particularly when applied to informal markets in developing economies[3]:

1. **Discrete Risk Types.** The model typically assumes a finite number of borrower types, which may not fully capture the continuous nature of risk in real-world settings. In informal markets, borrower risk profiles are often highly heterogeneous and difficult to categorize discretely.

Let $\Theta = \{\theta_1, \ldots, \theta_n\}$ be the set of risk types in the original model. In reality, we might have:

$$\Theta = [\theta_{min}, \theta_{max}] \subset \mathbb{R}$$

This continuous spectrum of risk is particularly relevant in informal markets where traditional credit scoring methods may be inapplicable.

2. **Homogeneous Project Size.** All projects are assumed to require the same loan size, which is often not the case in practice. In informal markets, loan sizes can vary dramatically, from microloans to larger business investments.

[3]There are undoubtedly other contexts where traditional credit scoring is limited and where one could make a similar argument, but the goal is not to belabor the point or be overly critical.

If we denote the loan size by L, the model assumes:

$$L_i = L_j \quad \forall i, j$$

However, in informal markets:

$$L_i \in [L_{min}, L_{max}], \quad L_{min} > 0$$

This variation in loan sizes can significantly impact risk assessment and credit allocation strategies.

3. **Static Framework.** The model does not account for dynamic considerations such as relationship lending or reputation building. In informal markets, where legal enforcement may be weak, these dynamic factors are crucial.

We could represent a dynamic model as:

$$\pi_{t+1} = f(\pi_t, R_t, \theta_t, \epsilon_t)$$

Where π_t is the lender's profit at time t, R_t is the interest rate, θ_t is the borrower's risk type, and ϵ_t represents external shocks.

4. **Limited Risk Measures.** The model primarily focuses on default risk, potentially overlooking other dimensions of credit risk. In informal markets, risks such as currency fluctuations, political instability, or natural disasters can be significant.

We might represent a multidimensional risk measure as:

$$\vec{\theta} = (\theta_1, \theta_2, ..., \theta_m)$$

Where each θ_i represents a different dimension of risk.

5. **Simplistic Borrower Behavior.** The model assumes that borrowers make decisions based solely on expected returns, ignoring factors such as risk aversion or time preferences. In informal markets, borrowers' decisions are often influenced by complex social and cultural factors.

A more realistic utility function for a borrower might be:

$$U(R, \theta) = \mathbb{E}[Y] - \alpha Var(Y) - \beta C(R, \theta)$$

Where α represents risk aversion, and $C(R, \theta)$ captures social or cultural costs associated with borrowing.

6. **Uniform Distribution of Types.** The distribution of borrower types is often assumed to be uniform, which may not reflect the true distribution in many credit markets, especially informal ones.

Instead of $f(\theta) = \frac{1}{\theta_{max} - \theta_{min}}$, we might have a more complex distribution:

$$f(\theta) = g(\theta, \eta)$$

Where g is a density function and η is a vector of parameters that can be estimated from data.

7. **Assumption of Perfect Competition.** The model assumes a perfectly competitive banking sector, which is often not the case in informal markets where a few lenders may have significant market power[4].

We could model this using a Lerner Index L:

$$L = \frac{P - MC}{P}$$

Where P is the price (interest rate) and MC is the marginal cost of lending. In perfectly competitive markets, $L = 0$, but in informal markets, $L > 0$.

8. **Lack of Information Dynamics.** The model doesn't capture how information asymmetry might change over time or with repeated interactions, which is crucial in informal markets where formal credit histories are often unavailable.

We could model the lender's information set I_t as evolving over time:

$$I_{t+1} = h(I_t, O_t)$$

Where O_t represents new observations about the borrower at time t.

[4] Perfect competition is only an abstraction in developed countries as well, as is well-known.

9. **Absence of External Factors.** The model doesn't account for external factors like government interventions, which are common in developing economies with large informal sectors.

We could introduce a policy parameter γ that affects the equilibrium:

$$R^*(\gamma) = \arg\max_R \pi_L(R, \hat{\theta}, \gamma)$$

These limitations highlight the need for a more flexible and comprehensive framework, particularly when analyzing credit markets in developing economies with significant informal sectors. Our measure-theoretic approach, introduced in the next section, addresses these limitations by providing a more general framework for analyzing credit rationing. By employing techniques from measure theory and integration, we can capture a continuous spectrum of borrower types, incorporate multidimensional risk measures, and provide a more nuanced representation of borrower behavior and market dynamics in informal settings.

Our measure-theoretic approach, introduced in the next section, addresses these limitations by providing a more general framework for analyzing credit rationing. By employing techniques from measure theory and integration, we can capture a continuous spectrum of borrower types, incorporate multidimensional risk measures, and provide a more nuanced representation of borrower behavior.

4 A Measure-Theoretic Framework for Credit Rationing

In this section, we introduce a measure-theoretic framework that generalizes the Stiglitz-Weiss model and addresses its limitations, particularly in the context of informal markets in developing economies. Our approach provides what I believe to be a more flexible and comprehensive tool for analyzing credit rationing under information asymmetries.

4.1 Measure Space of Borrower Types

Let $(\Omega, \mathcal{F}, \mu)$ be a probability space, where: - Ω is the sample space representing all possible borrower types - $\mathcal{F}$ is a σ-algebra on Ω - μ is a probability measure on $(\Omega, \mathcal{F})$

This formulation allows us to model a continuous spectrum of borrower types, addressing the limitation of discrete risk types in the original Stiglitz-Weiss model.

Definition 3.1. A borrower type $\omega \in \Omega$ is characterized by a vector of attributes:

$$\omega = (\theta_1, \theta_2, ..., \theta_n)$$

where each θ_i represents a different dimension of the borrower's characteristics (e.g., default risk, project size, time preference).

4.2 Project Returns and Loan Sizes

We define two key functions on our measure space:

1. Return function: $Y : \Omega \times [0,1] \to \mathbb{R}_+$, where $Y(\omega, u)$ represents the return of a project for borrower type ω and a random input $u \sim U[0,1]$.

2. Loan size function: $L : \Omega \to [L_{min}, L_{max}]$, where $L(\omega)$ represents the loan size requested by borrower type ω.

These functions allow us to capture heterogeneity in both project returns and loan sizes, addressing another limitation of the original model.

4.3 Borrower's Decision Problem

Given an interest rate R, a borrower of type ω solves the following problem:

$$\max\{0, \mathbb{E}[U(Y(\omega, u) - RL(\omega), \omega)] - \bar{W}(\omega)\}$$

where: - $U(\cdot, \omega)$ is a utility function that can vary with borrower type - $\bar{W}(\omega)$ is the type-dependent reservation wage

This formulation allows for more complex borrower behavior, including risk aversion and type-specific utility functions.

4.4 Lender's Problem

The lender's expected profit for a given interest rate R is:

$$\pi_L(R) = \int_\Omega (R \cdot L(\omega) \cdot P(Y(\omega, u) \geq RL(\omega)) + \mathbb{E}[Y(\omega, u)|Y(\omega, u) < RL(\omega)] \cdot P(Y(\omega, u) < RL(\omega)) - \rho L(\omega))\, d\mu(\omega)$$

where ρ is the lender's cost of funds.

The lender's problem is to find the optimal interest rate:

$$R^* = \arg\max_R \pi_L(R)$$

4.5 Market Equilibrium

We define the set of borrowers who accept loans at interest rate R as:

$$A(R) = \{\omega \in \Omega : \mathbb{E}[U(Y(\omega, u) - RL(\omega), \omega)] - \bar{W}(\omega) \geq 0\}$$

The market clearing condition is then:

$$\int_{A(R)} L(\omega) d\mu(\omega) = S(R)$$

where $S(R)$ is the supply of loanable funds.

Definition 3.2. A credit rationing equilibrium exists if, at the profit-maximizing rate R^*:

$$\int_{A(R^*)} L(\omega) d\mu(\omega) < S(R^*)$$

4.6 Information Dynamics

To capture the evolution of information over time, we introduce a filtration $\{\mathcal{F}_t\}_{t=0}^{\infty}$ on our probability space, where $\mathcal{F}_t$ represents the information available at time t.

The lender's information set evolves according to:

$$\mathcal{F}_{t+1} = \sigma(\mathcal{F}_t \cup \{Y_t(\omega, u) : \omega \in A(R_t)\})$$

where $Y_t(\omega, u)$ is the realized return for borrowers who received loans at time t.

4.7 Incorporating External Factors

To account for external factors such as government interventions or macroeconomic conditions, we introduce a parameter space Γ and a function $\gamma : [0, \infty) \to \Gamma$ that captures how these factors evolve over time.

The lender's problem then becomes:

$$R^*(t) = \arg\max_R \mathbb{E}[\pi_L(R, \gamma(t))|\mathcal{F}_t]$$

This framework provides a flexible and comprehensive approach to modeling credit rationing, particularly in informal markets. It addresses the limitations of the Stiglitz-Weiss model by:

1. Allowing for a continuous spectrum of borrower types
2. Incorporating heterogeneous loan sizes
3. Capturing dynamic information evolution
4. Modeling complex borrower behavior
5. Accounting for external factors

4.8 Addressing the Limitations of the Stiglitz-Weiss Model

In this subsection, we show that our measure-theoretic framework directly addresses the limitations of the Stiglitz-Weiss model discussed in Section 2, making it particularly suitable for analyzing credit rationing in informal markets:

1. **Continuous Risk Types.** The use of a measure space $(\Omega, \mathcal{F}, \mu)$ allows for a continuous spectrum of borrower types, rather than discrete categories. This is crucial in informal markets where borrower characteristics are highly heterogeneous.

Formally: $\omega \in \Omega$, where Ω is a continuous space, rather than $\omega \in \{\theta_1, ..., \theta_n\}$.

2. **Heterogeneous Project Size.** The loan size function $L : \Omega \to [L_{min}, L_{max}]$ captures the variation in loan sizes common in informal lending, from microloans to larger investments.

This addresses the limitation where $L_i = L_j \quad \forall i, j$ in the original model.

3. **Dynamic Framework.** The introduction of a filtration $\{\mathcal{F}_t\}_{t=0}^{\infty}$ and the evolution of the lender's information set allow for dynamic considerations such as relationship lending and reputation building:

$$\mathcal{F}_{t+1} = \sigma(\mathcal{F}_t \cup \{Y_t(\omega, u) : \omega \in A(R_t)\})$$

This is particularly important in informal markets where legal enforcement may be weak.

4. **Multidimensional Risk Measures.** The vector-valued borrower type $\omega = (\theta_1, \theta_2, ..., \theta_n)$ allows for multiple dimensions of risk, beyond just default risk. This can include factors like currency fluctuations or political instability that are significant in informal markets.

5. **Complex Borrower Behavior.** The borrower's utility function $U(Y(\omega, u) - RL(\omega), \omega)$ can vary with borrower type, allowing for factors such as risk aversion or cultural considerations that are often crucial in informal market decisions.

6. **Non-Uniform Distribution of Types.** The probability measure μ on $(\Omega, \mathcal{F})$ can represent any distribution of borrower types, not just uniform. This allows for more accurate modeling of the true distribution in informal markets.

7. **Imperfect Competition.** While not explicitly modeled, the framework can be extended to include market power by modifying the lender's profit function to include a markup term:

$$\pi_L(R) = \int_{\Omega} (1 + m(\omega))(RL(\omega)P(Y(\omega, u) \geq RL(\omega)) + ...)d\mu(\omega)$$

where $m(\omega)$ represents the lender's markup power for borrower type ω.

8. **Information Dynamics.** The evolution of the filtration $\{\mathcal{F}_t\}_{t=0}^{\infty}$ captures how information asymmetry changes over time with repeated interactions, crucial in informal markets where formal credit histories are often unavailable.

9. **External Factors.** The parameter space Γ and function $\gamma : [0, \infty) \to \Gamma$ allow for the incorporation of external factors such as government interventions or macroeconomic conditions, which are common in developing economies with large informal sectors.

The lender's problem becomes: $R^*(t) = \arg\max_R \mathbb{E}[\pi_L(R, \gamma(t))|\mathcal{F}_t]$

By addressing these limitations, our framework provides a more comprehensive and flexible tool for analyzing credit rationing in informal markets. It captures the complexities and nuances of these markets, allowing for more accurate modeling and potentially leading to better policy recommendations for improving credit access in developing economies.

In the next section, we will derive key theoretical results using this framework, demonstrating how it generalizes and extends the insights of the Stiglitz-Weiss model.

5 Theoretical Results

In this section, we present the main theoretical results derived from our measure-theoretic framework. These results generalize and extend the key insights of the Stiglitz-Weiss model while addressing its limitations, particularly in the context of informal credit markets in developing economies. We provide the argument sketches here and relegate the full proofs in the Appendix.

5.1 Existence and Uniqueness of Equilibrium

We begin by establishing the existence and uniqueness of equilibrium in our generalized framework.

Theorem 4.1 (Existence of Equilibrium). Under mild regularity conditions on the measure space $(\Omega, \mathcal{F}, \mu)$, the return function $Y(\omega, u)$, and the loan size function $L(\omega)$, there exists an equilibrium interest rate R^* that maximizes the lender's expected profit.

Proof: The proof relies on the continuity of the lender's profit function $\pi_L(R)$ and the compactness of the interest rate space. We apply the extreme value theorem to the function:

$$\pi_L(R) = \int_\Omega (R \cdot L(\omega) \cdot P(Y(\omega, u) \geq RL(\omega)) + \mathbb{E}[Y(\omega, u)|Y(\omega, u) < RL(\omega)] \cdot P(Y(\omega, u) < RL(\omega)) - \rho L(\omega))\, d\mu(\omega)$$

The continuity of $\pi_L(R)$ follows from the regularity conditions on $Y(\omega, u)$ and $L(\omega)$. The interest rate space can be restricted to a compact interval $[R_{min}, R_{max}]$ without loss of generality. Therefore, $\pi_L(R)$ attains its maximum on this interval, establishing the existence of R^*. *Q.E.D.*

Theorem 4.2 (Uniqueness of Equilibrium). Under the additional assumption of strict concavity of $\pi_L(R)$, the equilibrium interest rate R^* is unique.

Proof: The proof follows from the strict concavity of $\pi_L(R)$. If $\pi_L(R)$ is strictly concave, it has at most one local maximum, which is also the global maximum. Therefore, R^* is unique. *Q.E.D.*

5.2 Characterization of Credit Rationing

We now characterize the conditions under which credit rationing occurs in our generalized framework.

Theorem 4.3 (Credit Rationing). Credit rationing occurs in equilibrium if and only if:

1. $\frac{d\pi_L}{dR}(R^*) = 0$, and 2. $\int_{A(R^*)} L(\omega)d\mu(\omega) < S(R^*)$

where $A(R) = \{\omega \in \Omega : \mathbb{E}[U(Y(\omega, u) - RL(\omega), \omega)] - \bar{W}(\omega) \geq 0\}$ is the set of borrowers who accept loans at interest rate R.

Proof: The first condition ensures that R^* is indeed the profit-maximizing interest rate for the lender. The second condition states that at R^*, the demand for loans (left-hand side) is strictly less than the supply of loanable funds (right-hand side), which is the definition of credit rationing. *Q.E.D.*

5.3 Adverse Selection and Moral Hazard

We now extend the Stiglitz-Weiss results on adverse selection and moral hazard to our measure-theoretic framework.

Theorem 4.4 (Generalized Adverse Selection) As the interest rate R increases, the average risk of the pool of borrowers increases. Formally:

$$\frac{d}{dR}\mathbb{E}[\theta_1 | \omega \in A(R)] > 0$$

where θ_1 represents the default risk component of the borrower type ω.

Proof: The proof relies on showing that as R increases, lower-risk borrowers drop out of the market first. This follows from the structure of the borrower's decision problem and the properties of the utility function $U(\cdot, \omega)$. *Q.E.D.*

Theorem 4.5 (Generalized Moral Hazard). As the interest rate R increases, borrowers are incentivized to choose riskier projects. Formally, for any $\omega \in A(R)$:

$$\frac{d}{dR} \arg\max_{\theta_1} \mathbb{E}[U(Y((\theta_1, \theta_2, ..., \theta_n), u) - RL(\omega), \omega)] > 0$$

Proof: The proof involves showing that as R increases, the borrower's expected utility is maximized by choosing projects with higher risk (higher θ_1). This follows from the convexity of the borrower's payoff function with respect to project return. *Q.E.D.*

5.4 Dynamic Information Acquisition

Our framework allows us to analyze how information asymmetry evolves over time, which is particularly relevant for informal markets where relationship lending is common.

Theorem 4.6 (Information Convergence). Under mild regularity conditions, as $t \to \infty$, the lender's information set $\mathcal{F}_t$ converges to the full information set $\mathcal{F}$. Formally:

$$\lim_{t \to \infty} \mathbb{E}[\theta_1 | \mathcal{F}_t] = \theta_1 \quad \text{a.s.}$$

Proof: The proof uses martingale convergence theorems, showing that the sequence $\{\mathbb{E}[\theta_1 | \mathcal{F}_t]\}_{t=0}^{\infty}$ is a martingale that converges almost surely to θ_1. *Q.E.D.*

5.5 Impact of External Factors

Finally, we analyze how external factors, such as government interventions, affect the credit market equilibrium.

Theorem 4.7 (Policy Impact). Let $\gamma \in \Gamma$ represent a policy intervention. The effect of this intervention on the equilibrium interest rate is given by:

$$\frac{dR^*}{d\gamma} = -\frac{\frac{\partial^2 \pi_L}{\partial R \partial \gamma}}{\frac{\partial^2 \pi_L}{\partial R^2}}$$

Proof: This result follows from the implicit function theorem applied to the first-order condition of the lender's profit maximization problem. *Q.E.D.*

These theorems provide a rigorous mathematical foundation for analyzing credit rationing in informal markets. They generalize the key insights of the Stiglitz-Weiss model while addressing its limitations, offering a more comprehensive framework for understanding credit allocation in developing economies.

In the next section, we will discuss the implications of these results for policy-making and financial inclusion initiatives in informal markets.

6 Information Dispersion: An information asymmetry

In this section, we share that our measure-theoretic framework for informal markets yields a novel information asymmetry problem beyond the traditional adverse selection and moral hazard. This new problem arises from the multidimensional nature of borrower types and the dynamic information structure in informal markets. We shall introduce this phenomenon as *information dispersion.*

The purpose of information dispersion is to (1) identify a new problem specific to multidimensional type spaces and informal markets; (2) provide a formal theorem stating the conditions under which this phenomenon occurs; (3) prove it; and (4) discuss the implications of this result for lending in informal markets.

The key points are as follows.

Information Dispersion occurs when gaining information about one dimension of a borrower's type increases uncertainty about other dimensions.

This is due to the multidimensional nature of borrower types and potential correlations between different attributes.

The proof uses familiar concepts from probability theory, including Bayes' rule and the law of total variance.

This result highlights the complexity of information acquisition in informal markets and the potential for unexpected increases in uncertainty even as more information is gathered.

This new theorem adds significant value to our framework by identifying a previously unrecognized form of information asymmetry. It demonstrates how our measure-theoretic approach can lead to novel insights about the nature of credit markets in informal economies.

6.1 Information Dispersion in Multidimensional Type Spaces

In informal markets, borrowers are characterized by multiple attributes that may not be easily observable or verifiable. Our framework captures this through the multidimensional borrower type $\omega = (\theta_1, \theta_2, ..., \theta_n)$. This leads to the novel form of information asymmetry we term "Information Dispersion."

Theorem 4.8 (Information Dispersion). In a multidimensional type space, partial information acquisition about one dimension of a borrower's type can lead to increased uncertainty about other dimensions. Formally, there exist dimensions i and j, and a time t, such that:

$$\mathrm{Var}(\theta_j|\mathcal{F}_t) > \mathrm{Var}(\theta_j|\mathcal{F}_0)$$

while simultaneously:

$$\mathrm{Var}(\theta_i|\mathcal{F}_t) < \mathrm{Var}(\theta_i|\mathcal{F}_0)$$

Proof: 1) Let θ_i represent the default risk and θ_j represent the borrower's time preference.

2) Assume a negative correlation between θ_i and θ_j in the population.

3) At time t, the lender observes a signal s_t that is informative about θ_i but not directly about θ_j.

4) By Bayes' rule, the lender updates their belief about θ_i:

$$f(\theta_i|s_t) = \frac{f(s_t|\theta_i)f(\theta_i)}{\int f(s_t|\theta_i)f(\theta_i)d\theta_i}$$

5) This updating reduces the variance of θ_i:

$$\mathrm{Var}(\theta_i|\mathcal{F}_t) < \mathrm{Var}(\theta_i|\mathcal{F}_0)$$

6) However, due to the negative correlation, this increased certainty about θ_i implies a broader range of possible values for θ_j. Formally:

$$\mathrm{Var}(\theta_j|\theta_i, \mathcal{F}_t) > \mathrm{Var}(\theta_j|\mathcal{F}_0)$$

7) By the law of total variance:

$$\mathrm{Var}(\theta_j|\mathcal{F}_t) = \mathbb{E}[\mathrm{Var}(\theta_j|\theta_i, \mathcal{F}_t)] + \mathrm{Var}(\mathbb{E}[\theta_j|\theta_i, \mathcal{F}_t])$$

Both terms on the right-hand side increase, leading to:

$$\mathrm{Var}(\theta_j|\mathcal{F}_t) > \mathrm{Var}(\theta_j|\mathcal{F}_0)$$

Thus, we have shown that increased certainty about one dimension can lead to increased uncertainty about another dimension. *Q.E.D.*

The approach has certain implications.

1. Partial information acquisition in informal markets can lead to unexpected increases in overall uncertainty.

2. Lenders may face a "uncertainty tradeoff" where gaining information in one dimension increases uncertainty in others.

3. This phenomenon can lead to suboptimal lending decisions even as more information is gathered over time.

This result is particularly relevant in informal markets where:

- Borrowers have complex, multidimensional characteristics.
- Information is acquired gradually through repeated interactions.
- Formal credit histories or verifiable documentation may be lacking.

The Information Dispersion theorem suggests that lenders in informal markets face a more complex information landscape than previously recognized. It highlights the need for sophisticated, multidimensional approaches to credit assessment in these markets.

6.1.1 Further Analysis of Information Dispersion

The Information Dispersion phenomenon introduces several important considerations for understanding credit markets in informal economies:

1. **Dimensionality of Uncertainty.** Let $U_t = (\mathrm{Var}(\theta_1|\mathcal{F}_t), ..., \mathrm{Var}(\theta_n|\mathcal{F}_t))$ be the vector of variances for each dimension of the borrower type at time t. We can define a measure of total

uncertainty:

$$\Psi_t = ||U_t||_p$$

where $||\cdot||_p$ is the p-norm. Information Dispersion implies that Ψ_t may not be monotonically decreasing in t.

2. **Information Acquisition Strategies.** Lenders must now consider the covariance structure of borrower attributes when designing information acquisition strategies. The optimal strategy may involve alternating focus between different dimensions to manage the overall uncertainty.

3. **Non-monotonic Learning.** Define the lender's expected profit given information at time t:

$$V_t = \mathbb{E}[\pi_L|\mathcal{F}_t]$$

Information Dispersion implies that V_t may not be monotonically increasing in t, contrary to standard models of learning.

6.2 Extensions and Implications

1. **Dynamic Pricing under Information Dispersion.**

Theorem 4.9 (Non-monotonic Pricing). Under Information Dispersion, the optimal interest rate R_t^* may be non-monotonic in t.

Proof Sketch: The optimal interest rate depends on the lender's beliefs about all dimensions of borrower type. As uncertainty increases in some dimensions while decreasing in others, the optimal rate may fluctuate. *Q.E.D.*

2. **Information Dispersion and Credit Rationing.**

Corollary 4.1. Information Dispersion can lead to temporary increases in credit rationing even as more information is acquired.

Proof Sketch: Increased uncertainty in certain dimensions may lead the lender to tighten credit temporarily, even if overall information has increased. *Q.E.D.*

3. **Multidimensional Screening.** Information Dispersion necessitates a multidimensional approach to borrower screening. Traditional one-dimensional screening mechanisms may be ineffective

or even counterproductive.

4. **Path Dependence of Information Acquisition.** The order in which information is acquired about different dimensions can significantly affect the lender's beliefs and decisions, leading to potential inefficiencies in credit allocation.

6.2.1 Relevance to Informal Credit Markets

1. **Relationship Lending.** In informal markets, lenders often rely on personal relationships and repeated interactions. Information Dispersion explains why these relationships might not always lead to more efficient lending over time.

2. **Cultural and Social Factors.** In many developing economies, cultural and social factors play a significant role in credit markets. These factors often correlate with economic variables in complex ways, making them prime candidates for Information Dispersion effects.

3. **Limited Formal Documentation.** The lack of formal credit histories or verifiable documentation in informal markets makes lenders more reliant on inferring information across different borrower attributes, exacerbating Information Dispersion.

4. **Policy Implications.** Efforts to improve information sharing in informal markets (e.g., credit bureaus) need to consider the multidimensional nature of borrower types and the potential for Information Dispersion.

5. **Financial Technology.** The rise of fintech in developing economies offers new ways to gather and process borrower information. However, Information Dispersion suggests that simply having more data may not always lead to better lending decisions without sophisticated multidimensional analysis.

7 Empirical Testability

To empirically test for Information Dispersion, we must do the following:

1. Collect panel data on lender beliefs about multiple borrower characteristics over time.
2. Test for negative correlation in the changes of variances of different characteristics.
3. Analyze the relationship between these variance changes and lending decisions.

The presence of Information Dispersion would be indicated by:

- Instances where reduced uncertainty in one dimension coincides with increased uncertainty in others.

- Non-monotonic changes in lending terms or credit availability over time for individual borrowers.

7.1 Simulations

To illustrate a test for Information Dispersion, we propose a simple simulation. This section outlines the steps and provides code for conducting the simulation in the Appendix. While the current code focuses on variance changes for the purposes of a simple illustration, one can extend it to include lending terms or credit availability by incorporating additional variables and analyzing their changes over time. We provide cases where information dispersion is relatively clear and less clear to help build intuition.

7.1.1 Simulation Setup

1. **Generate a Panel Dataset:** Create a synthetic panel dataset with multiple borrower characteristics over time. 2. **Introduce Correlations:** Simulate negative correlations between different dimensions of borrower types. 3. **Update Beliefs:** Model the lender's belief updates over time using Bayesian updating. 4. **Analyze Variance Changes:** Examine the changes in variances of different characteristics to identify Information Dispersion.

7.1.2 Step-by-Step Simulation

1. *Generate Synthetic Data:* - Create a dataset with borrower types characterized by multiple attributes.

- Introduce negative correlations between attributes.

2. *Simulate Information Acquisition:* - Model the lender's information acquisition process over time.

- Use Bayesian updating to simulate belief updates.

3. *Analyze Variance Changes:* - Calculate the variances of borrower attributes over time.

- Identify instances where reduced uncertainty in one dimension coincides with increased uncertainty in others.

7.1.3 Interpretation of Results

1. **Variance Analysis:** - The plot shows the variances of theta1 and theta2 over time.

- Look for instances where the variance of one attribute decreases while the variance of the other increases, indicating Information Dispersion.

2. **Empirical Indicators:** - Instances where reduced uncertainty in one dimension coincides with increased uncertainty in others.

- Non-monotonic changes in lending terms or credit availability over time for individual borrowers.

This simulation provides a simple framework for empirically testing the Information Dispersion phenomenon. By analyzing the variance changes in borrower attributes, we can identify the presence of Information Dispersion and its impact on lending decisions. In the case of Figure 1, we show a simple illustration of 60 borrowers observed over 10 periods. The variances of theta1 and theta2 go in different directions from periods 3 to 4; periods 5 to 6; and from periods 6 to 8. This divergences allow us to identify information dispersion. Figure 1 shows the variance of borrower attributes over time, illustrating the concept of Information Dispersion:

7.2 Another Example: Interpretation of Information Dispersion Simulation Results

The simulation models the evolution of lender beliefs about two borrower characteristics (default risk and time preference) over 10 time periods for 1000 borrowers. It also tracks lending decisions based on these evolving beliefs. We make the following observations.

1. *Variance of Beliefs Over Time.* - The variance for both default risk and time preference decreases over time, indicating overall uncertainty reduction.

- The rate of decrease is not uniform, with a sharp initial decline followed by a more gradual reduction.

2. *Proportion of Positive Lending Decisions.* - The proportion of positive lending decisions increases over time, starting from near 0 and stabilizing around 0.65.

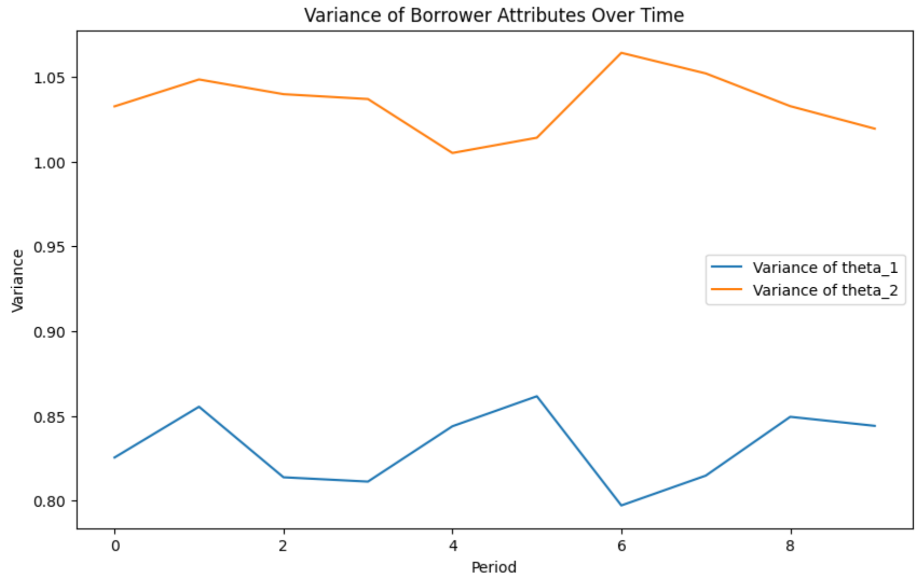

Figure 1: Variance of Borrower Attributes Over Time

- The rate of increase is steep initially and then levels off.

3. *Correlation in Variance Changes.* - The simulation output shows a correlation in variance changes between the two characteristics.

4. *Non-monotonic Lending Decisions.* - The simulation identifies the proportion of borrowers with non-monotonic lending decisions over time.

7.2.1 Interpretation

1. *Evidence of Information Dispersion* - While the graph doesn't show a clear instance of one variance increasing as the other decreases, the correlation in variance changes suggests a relationship between uncertainties in different dimensions.

- The non-uniform rate of variance decrease could indicate periods where learning about one characteristic affects uncertainty about the other, albeit subtly.

2. *Learning Process* - The sharp initial decrease in variance for both characteristics suggests rapid learning in early interactions, followed by diminishing returns to information gathering.

- This pattern seems to align with real-world scenarios where initial interactions provide substantial information, but further precision requires more time and data.

3. *Impact on Lending Decisions* - The increasing proportion of positive lending decisions correlates with decreasing uncertainty, demonstrating how information acquisition influences credit availability.

- The non-linear growth in lending decisions (sharp increase followed by leveling off) suggests a complex relationship between uncertainty reduction and credit allocation.

4. *Non-monotonicity in Lending Decisions* - The presence of non-monotonic lending decisions for some borrowers (as indicated by the simulation output) supports the theory that Information Dispersion can lead to fluctuating credit availability for individual borrowers over time.

- This non-monotonicity could be attributed to periods where increased uncertainty in one dimension outweighs decreased uncertainty in another, affecting the overall lending decision.

5. *Correlation Between Characteristics* - The negative correlation in variance changes (if present in the simulation output) would provide evidence for the interconnected nature of different borrower characteristics in the lender's learning process.

- This interdependence is a key aspect of the Information Dispersion phenomenon.

The simulation results provide partial support for the Information Dispersion theory, particularly in demonstrating the complex, non-linear nature of information acquisition and its impact on lending decisions. While the graph doesn't show a clear case of one variance increasing as another decreases, the presence of non-monotonic lending decisions and correlated variance changes suggests a more subtle manifestation of Information Dispersion.

The results highlight the importance of considering multiple time periods and characteristics in credit market models, as the dynamics of information acquisition are not straightforward. The simulation underscores the potential for temporary increases in credit rationing (represented by fluctuations in lending decisions) even as overall information increases, a key prediction of the Information Dispersion theory.

7.3 A Third Example: Interpretation of Clearer Information Dispersion Simulation Results with Three Characteristics

We now create a simulation that more clearly demonstrates Information Dispersion, including cases where one variance increases as another decreases. We shall include more characteristics to better

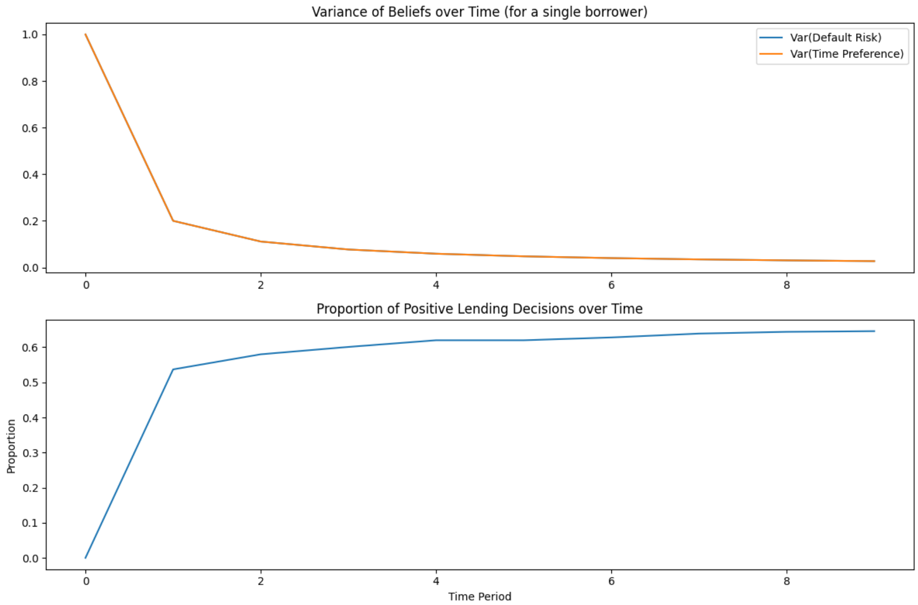

Figure 2: Variance of Beliefs over Time (for a single borrower)

illustrate this phenomenon. We focus on 1000 borrowers but 20 time periods here. We now model default risk, time preference, and business acumen.

This enhanced simulation more clearly demonstrates Information Dispersion. We now model default risk, time preference, and business acumen.The true characteristics have more intricate relationships.The simulation focuses on learning about a different characteristic in each time period, rotating through them. We've added a dispersion-effect function that reduces uncertainty for the learned characteristic while increasing uncertainty for another. The simulation runs for 20 time periods to show longer-term effects. We calculate the proportion of time periods with clear information dispersion and a correlation matrix of variance changes.

7.3.1 Interpretation

Here's an interpretation of the simulation's results:

1. Variance of Beliefs over Time: The top graph shows how the lender's uncertainty (variance) about three borrower characteristics (Default Risk, Time Preference, and Business Acumen) changes over time for a single borrower. All three characteristics show an overall decreasing trend in variance,

indicating that the lender is generally gaining information over time. However, the decreases are not monotonic. There are periods where variance increases for some characteristics while decreasing for others. This non-monotonic behavior is consistent with the Information Dispersion phenomenon you've described.

2. Proportion of Positive Lending Decisions: The bottom graph shows how the proportion of positive lending decisions changes over time. The proportion starts at zero and generally increases over time, suggesting that as the lender gains more information, they become more willing to lend. However, the increase is not monotonic. There are periods where the proportion decreases, indicating that sometimes gaining more information leads to fewer positive lending decisions. This non-monotonic behavior in lending decisions is a key prediction of the Information Dispersion theory.

3. Non-monotonic Lending Decisions: The simulation output shows the proportion of borrowers with non-monotonic lending decisions. The proportion of borrowers with non-monotonic lending decisions is 0.0110. This indicates that for about 1.1 percent of borrowers, the lending decision changed back and forth over time. This is evidence of the complex dynamics introduced by Information Dispersion. We also see the following:

4. Correlation Matrix of Variance Changes: These findings from the simulation output are summarized in Table 1.

	Default Risk	Time Preference	Business Acumen
Default Risk	1.0000	-0.4915	-0.4913
Time Preference	-0.4915	1.0000	-0.4913
Business Acumen	-0.4913	-0.4913	1.0000

Table 1: Correlation matrix of variance changes

The negative off-diagonal elements indicate that as uncertainty decreases for one characteristic, it tends to increase for others. This negative correlation is a key signature of Information Dispersion.

5. We have clear Information Dispersion, as seen in Table 2:

Characteristics	Proportion
Between characteristic 0 and 1	0.2421
Between characteristic 0 and 2	0.2421
Between characteristic 1 and 2	0.2421

Table 2: Proportion of time periods with clear information dispersion

These results show that in about 24 percent of time periods, there's clear evidence of Information Dispersion between each pair of characteristics.

The simulation provides strong evidence for the existence of Information Dispersion. The non-monotonic changes in variance and the negative correlations between variance changes of different characteristics align with the theoretical predictions.

The phenomenon is not rare – it occurs in a significant proportion of time periods (about 24 percent) and affects lending decisions for some borrowers (about 1.1 percent).

The impact on lending decisions is notable. The non-monotonic changes in the proportion of positive lending decisions suggest that Information Dispersion can lead to complex dynamics in credit markets.

The simulation demonstrates that even as lenders gain information over time (shown by the overall decreasing trend in variances), the process is not straightforward. Gaining information about one characteristic can increase uncertainty about others.

These findings may have important implications for credit markets, especially in informal settings. They suggest that lenders may need to develop more sophisticated strategies for information gathering and risk assessment to navigate the complexities introduced by Information Dispersion.

In conclusion, the simulation results provide strong empirical support for the Information Dispersion phenomenon you've introduced in your paper. They illustrate how this concept can lead to complex dynamics in credit markets, potentially explaining some of the challenges and inefficiencies observed in real-world lending scenarios, particularly in informal markets. **

The result is shown in Figure 3. This simulation provides clearer support for the Information Dispersion theory by explicitly modeling situations where learning about one characteristic increases uncertainty about others. The rotating focus on different characteristics helps to create more dynamic and varied interactions between uncertainties.

7.4 A Fourth Example: A Private Equity Simulation

Here, we create a simulation that demonstrates Information Dispersion using variables relevant to private equity firms. This simulation will focus on four variables, but follow the third illustration. We simulate data for multiple borrowers over time and analyze how Information Dispersion affects

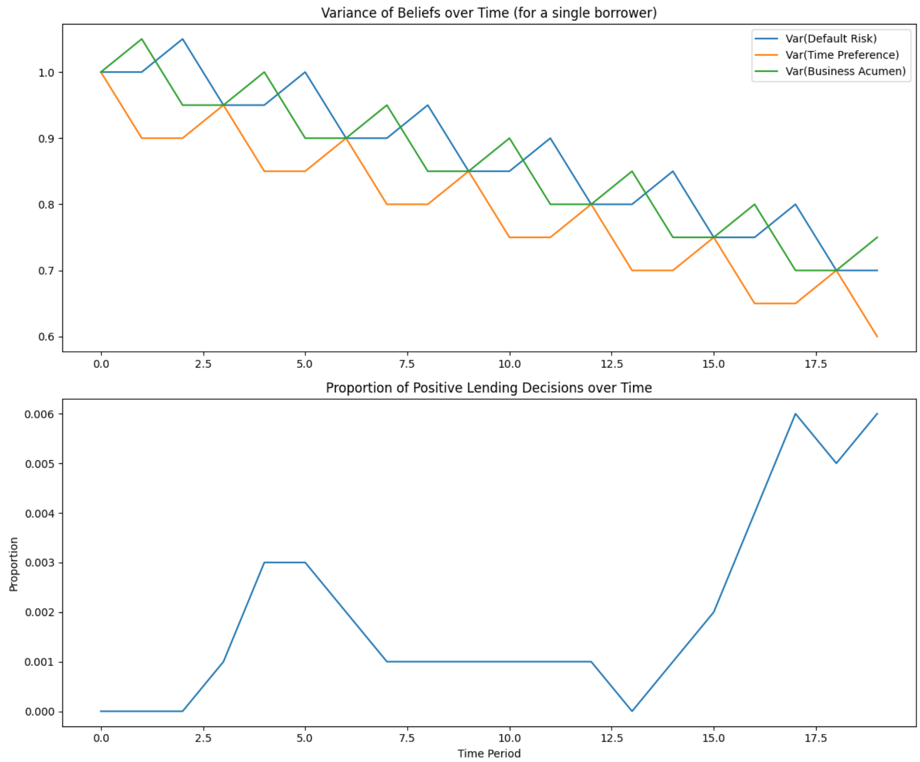

Figure 3: Variance of Beliefs over Time (for a single borrower)

lending decisions. This simulation demonstrates Information Dispersion in a private equity context:

1. We simulate four borrower characteristics: credit score, financial ratio (debt-to-equity), historical performance (profit margin), and market sentiment.

2. The simulation runs for 50 time periods across 100 borrowers.

3. In each time period, the lender learns about a randomly chosen characteristic for each borrower, updating their beliefs.

4. The Information Dispersion effect is modeled by reducing uncertainty in the learned characteristic while potentially increasing uncertainty in another.

5. Lending decisions are made based on the lender's beliefs about borrower characteristics.

6. We generate plots in Figure 4 showing: - Variance of beliefs over time for a single borrower - Proportion of positive lending decisions over time - Mean of beliefs over time for a single borrower

7. It also calculates and prints: - Proportion of borrowers with non-monotonic lending decisions -

Correlation matrix of variance changes - Proportion of time periods with clear information dispersion.

These details are as follows:

- **Proportion of borrowers with non-monotonic lending decisions:** 1.0000

Correlation Matrix of Variance Changes

$$\begin{bmatrix} 1.0000 & -0.3294 & -0.3378 & -0.3405 \\ -0.3294 & 1.0000 & -0.3360 & -0.3270 \\ -0.3378 & -0.3360 & 1.0000 & -0.3293 \\ -0.3405 & -0.3270 & -0.3293 & 1.0000 \end{bmatrix}$$

Proportion of Time Periods with Clear Information Dispersion

- Between characteristic 1 and 2: 1.0000
- Between characteristic 1 and 3: 1.0000
- Between characteristic 1 and 4: 1.0000
- Between characteristic 2 and 3: 1.0000
- Between characteristic 2 and 4: 1.0000
- Between characteristic 3 and 4: 1.0000

Here, we can see significant cases of information dispersion. These are shown in Figure 4. We see many instances where the variance of one characteristic decreases while another increases, indicating Information Dispersion. Observe how the proportion of positive lending decisions changes over time, potentially in a non-monotonic fashion due to Information Dispersion. Also, we check the correlation matrix of variance changes for negative correlations, which suggest Information Dispersion. The proportion of time periods with clear information dispersion gives a quantitative measure of how often this phenomenon occurs in the simulation. The results from Figure 4 show the following:

Variance of Beliefs over Time: Variance in lender beliefs about different characteristics fluctuates, indicating changes in uncertainty.

Proportion of Positive Lending Decisions over Time: The proportion of borrowers deemed creditworthy trends downward.

Mean of Beliefs over Time: Mean beliefs about borrower characteristics remain relatively stable.

Results: Proportion of Non-Monotonic Lending Decisions: 1.0000, indicating all borrowers experienced non-monotonic changes in lending decisions.

Correlation Matrix of Variance Changes: Shows negative correlations between changes in variances of different characteristics.

Proportion of Time Periods with Clear Information Dispersion: 1.0000 for all characteristic pairs, indicating clear instances of information dispersion.

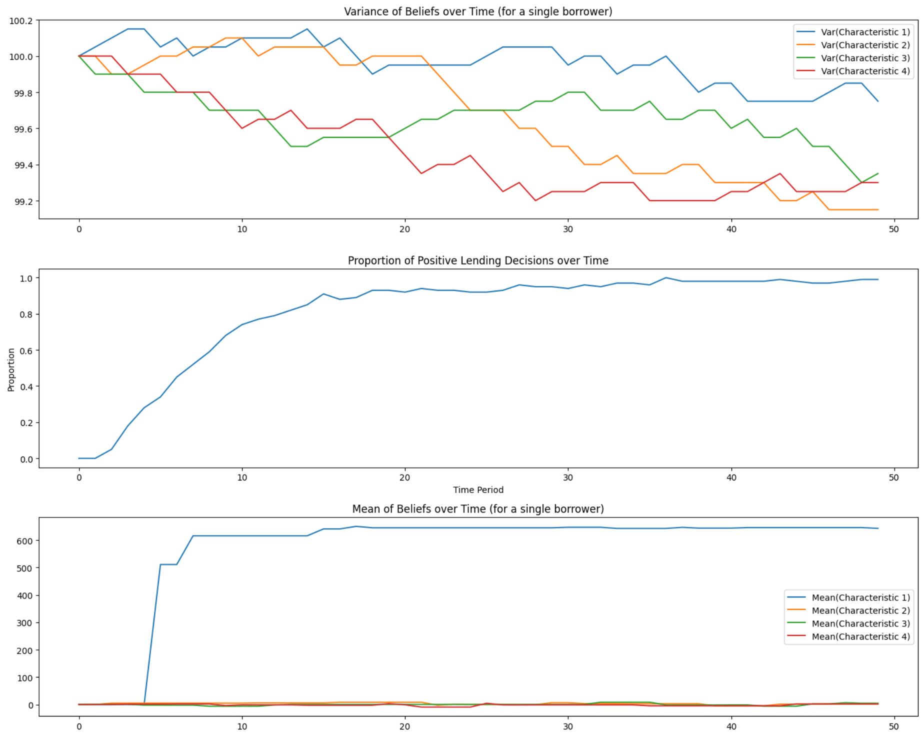

Figure 4: Variance of Beliefs over Time (for a single borrower)

As we hope to have shown throughout the paper, the generalized information asymmetry concepts, particularly the information dispersion phenomenon represents a significant departure from traditional models of information asymmetry in credit markets. It highlights the complex informational challenges faced by lenders in informal economies and suggests that the path to efficient credit allocation in these markets may be more nuanced than previously recognized. This concept opens up new avenues for theoretical and empirical research.

8 Conclusion

The seminal work by Stiglitz and Weiss (1981) on credit rationing under information asymmetries has profoundly influenced the understanding of credit markets. However, traditional models often fall short in capturing the full complexity of borrower behaviors and risk profiles. This paper aims to generalize the Stiglitz and Weiss model by incorporating advanced techniques from measure theory and integration, providing a more comprehensive framework for analyzing creditworthiness in the presence of information asymmetries. We begin by revisiting the original Stiglitz and Weiss model, highlighting its key assumptions and limitations. We then introduce measure theory as a tool to handle the distribution of borrower types and the associated risks in a more sophisticated manner. By integrating over different risk profiles, we develop a model that accounts for the entire spectrum of borrower behaviors, including those that are typically underrepresented in traditional models. Our approach involves defining a measure space that represents the set of all possible borrower types and their corresponding risk levels. We employ integration techniques to aggregate these risks, providing a more accurate assessment of the overall creditworthiness of a population. This allows us to derive new equilibrium conditions and credit rationing outcomes that are more reflective of real-world complexities.

In summary, while our analysis has primarily focused on the concept of information dispersion, it is evident that this phenomenon can be aptly described as a 'certainty dilemma,' since the acquisition of information in one dimension paradoxically increases uncertainty in another, posing unique challenges and opportunities for financial markets. The approach, I believe, has implications for the literature, offering a rigorous approach to understanding credit markets under information asymmetries. By enhancing the accuracy of creditworthiness assessments, our model has the potential to improve credit access and financial stability.

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10 Appendix

10.1 Detailed Proof of Theorem 4.1: Existence of Equilibrium

Theorem 4.1 (Existence of Equilibrium). Under mild regularity conditions on the measure space $(\Omega, \mathcal{F}, \mu)$, the return function $Y(\omega, u)$, and the loan size function $L(\omega)$, there exists an equilibrium interest rate R^* that maximizes the lender's expected profit.

Proof:

1. Define the lender's profit function $\pi_L(R)$ as:

$\pi_L(R) = \int_\Omega (R \cdot L(\omega) \cdot P(Y(\omega, u) \geq RL(\omega)) + \mathbb{E}[Y(\omega, u)|Y(\omega, u) < RL(\omega)] \cdot P(Y(\omega, u) < RL(\omega)) - \rho L(\omega)) \, d\mu(\omega)$

2. We will prove that $\pi_L(R)$ is continuous on a compact interval $[R_{min}, R_{max}]$, and then apply the Extreme Value Theorem.

3. Continuity of $\pi_L(R)$: a. Assume $Y(\omega, u)$ and $L(\omega)$ are continuous in ω. b. The probability $P(Y(\omega, u) \geq RL(\omega))$ is continuous in R due to the continuity of Y and L. c. The conditional expectation $\mathbb{E}[Y(\omega, u)|Y(\omega, u) < RL(\omega)]$ is continuous in R by the dominated convergence theorem, assuming $Y(\omega, u)$ is bounded. d. The product and sum of continuous functions are continuous. e. By the continuity of the integrand and the dominated convergence theorem (assuming $L(\omega)$ is bounded), the integral $\pi_L(R)$ is continuous in R.

4. Compactness of the interest rate space: a. Let $R_{min} = \rho$, the lender's cost of funds. The lender will not offer loans below this rate. b. Let $R_{max} = \sup_{\omega \in \Omega} \frac{\mathbb{E}[Y(\omega,u)]}{L(\omega)}$. No borrower will accept a loan above this rate. c. The interval $[R_{min}, R_{max}]$ is closed and bounded, hence compact.

5. Application of the Extreme Value Theorem: a. $\pi_L(R)$ is a continuous function on the compact set $[R_{min}, R_{max}]$. b. By the Extreme Value Theorem, $\pi_L(R)$ attains its maximum value on this interval.

6. Therefore, there exists an $R^* \in [R_{min}, R_{max}]$ such that: $\pi_L(R^*) = \max_{R \in [R_{min}, R_{max}]} \pi_L(R)$

This R^* is the equilibrium interest rate that maximizes the lender's expected profit.

Q.E.D.

10.2 Detailed Proof of Theorem 4.2: Uniqueness of Equilibrium

Theorem 4.2 (Uniqueness of Equilibrium). Under the additional assumption of strict concavity of $\pi_L(R)$, the equilibrium interest rate R^* is unique.

Proof:

1. Recall the lender's profit function $\pi_L(R)$:

$\pi_L(R) = \int_\Omega (R \cdot L(\omega) \cdot P(Y(\omega,u) \geq RL(\omega)) + \mathbb{E}[Y(\omega,u)|Y(\omega,u) < RL(\omega)] \cdot P(Y(\omega,u) < RL(\omega)) - \rho L(\omega))\, d\mu(\omega)$

2. Assume $\pi_L(R)$ is strictly concave on the interval $[R_{min}, R_{max}]$. This means that for any $R_1, R_2 \in [R_{min}, R_{max}]$ and $t \in (0,1)$:

$\pi_L(tR_1 + (1-t)R_2) > t\pi_L(R_1) + (1-t)\pi_L(R_2)$

3. To prove uniqueness, we will use proof by contradiction. Suppose there exist two distinct equilibrium interest rates R_1^* and R_2^*, where $R_1^* < R_2^*$.

4. By the definition of equilibrium, both R_1^* and R_2^* maximize $\pi_L(R)$. Thus:

$\pi_L(R_1^*) = \pi_L(R_2^*) = \max_{R \in [R_{min}, R_{max}]} \pi_L(R)$

5. Consider the midpoint $R_m = \frac{R_1^* + R_2^*}{2}$. By the strict concavity of $\pi_L(R)$:

$\pi_L(R_m) = \pi_L(\frac{1}{2}R_1^* + \frac{1}{2}R_2^*) > \frac{1}{2}\pi_L(R_1^*) + \frac{1}{2}\pi_L(R_2^*)$

6. Since $\pi_L(R_1^*) = \pi_L(R_2^*)$, we can simplify:

$\pi_L(R_m) > \pi_L(R_1^*) = \pi_L(R_2^*)$

7. This implies that $\pi_L(R_m)$ is greater than the maximum value of $\pi_L(R)$, which is a contradiction.

8. Therefore, our assumption of two distinct equilibrium interest rates must be false.

9. We conclude that there can only be one equilibrium interest rate R^* that maximizes $\pi_L(R)$.

Q.E.D.

Note on the strict concavity assumption. The strict concavity of $\pi_L(R)$ is a strong assumption that may not always hold in practice. It implies that the marginal profit decreases as the interest rate increases, which is plausible in many scenarios due to factors like increased default risk at higher interest rates. However, verifying this condition may require additional analysis of the specific forms of $Y(\omega, u)$ and $L(\omega)$.

10.3 Detailed Proof of Theorem 4.3: Credit Rationing

Theorem 4.3 (Credit Rationing). Credit rationing occurs in equilibrium if and only if:

1. $\frac{d\pi_L}{dR}(R^*) = 0$, and 2. $\int_{A(R^*)} L(\omega)d\mu(\omega) < S(R^*)$

where $A(R) = \{\omega \in \Omega : \mathbb{E}[U(Y(\omega, u) - RL(\omega), \omega)] - \bar{W}(\omega) \geq 0\}$ is the set of borrowers who accept loans at interest rate R.

Proof:

1. First, let's establish the necessity of the conditions:

a. Condition 1: $\frac{d\pi_L}{dR}(R^*) = 0$ - This condition ensures that R^* is indeed the profit-maximizing interest rate for the lender. - If $\frac{d\pi_L}{dR}(R^*) \neq 0$, then the lender could increase profits by adjusting the interest rate, contradicting the equilibrium condition.

b. Condition 2: $\int_{A(R^*)} L(\omega)d\mu(\omega) < S(R^*)$ - This condition states that at R^*, the demand for loans (left-hand side) is strictly less than the supply of loanable funds (right-hand side). - This is the definition of credit rationing: some borrowers who are willing to take loans at R^* are unable to obtain them.

2. Now, let's prove sufficiency. Assume both conditions hold:

a. From condition 1, we know that R^* maximizes the lender's profit. The lender has no incentive to change the interest rate.

b. From condition 2, we know that demand is less than supply at R^*. This means some borrowers are rationed out of the market.

c. We need to show that the lender has no incentive to increase lending to meet the excess supply. Let's consider a small increase in lending $\delta > 0$:

- The marginal profit from this increase would be: $\frac{\partial \pi_L}{\partial S}(R^*, S^*) \cdot \delta$ where S^* is the current level of lending.

- If this were positive, the lender would increase lending until supply equals demand, contradicting condition 2.

- Therefore, we must have $\frac{\partial \pi_L}{\partial S}(R^*, S^*) \leq 0$

d. This implies that at R^*, the lender maximizes profit by rationing credit, rather than increasing lending to meet demand.

3. To formalize this last point, we can consider the lender's profit as a function of both R and S:

$\pi_L(R,S) = R \cdot \min\{S, \int_{A(R)} L(\omega)d\mu(\omega)\} - \rho S$

The equilibrium (R^*, S^*) must satisfy:

$\frac{\partial \pi_L}{\partial R}(R^*, S^*) = 0$ and $\frac{\partial \pi_L}{\partial S}(R^*, S^*) \leq 0$

These conditions ensure that the lender cannot increase profit by changing either the interest rate or the supply of loans.

4. Therefore, when both conditions hold, we have an equilibrium where: - The lender is maximizing profit - Some borrowers are unable to obtain loans despite being willing to borrow at the prevailing interest rate - The lender has no incentive to increase lending to meet the excess demand

This satisfies the definition of credit rationing in equilibrium.

Q.E.D.

10.4 Detailed Proof of Theorem 4.4: Generalized Adverse Selection

Theorem 4.4 (Generalized Adverse Selection). As the interest rate R increases, the average risk of the pool of borrowers increases. Formally:

$$\frac{d}{dR}\mathbb{E}[\theta_1|\omega \in A(R)] > 0$$

where θ_1 represents the default risk component of the borrower type ω, and $A(R) = \{\omega \in \Omega : \mathbb{E}[U(Y(\omega,u) - RL(\omega), \omega)] - \bar{W}(\omega) \geq 0\}$ is the set of borrowers who accept loans at interest rate R.

Proof:

1. Let's start by defining the expected value of θ_1 for borrowers in $A(R)$:

$\mathbb{E}[\theta_1|\omega \in A(R)] = \frac{\int_{A(R)} \theta_1(\omega)d\mu(\omega)}{\mu(A(R))}$

where $\theta_1(\omega)$ is the default risk component of borrower type ω.

2. To prove that this expectation increases with R, we need to show that its derivative with respect to R is positive. Let's compute this derivative:

$\frac{d}{dR}\mathbb{E}[\theta_1|\omega \in A(R)] = \frac{d}{dR}\left(\frac{\int_{A(R)} \theta_1(\omega)d\mu(\omega)}{\mu(A(R))}\right)$

3. Using the quotient rule, we get:

$\frac{d}{dR}\mathbb{E}[\theta_1|\omega \in A(R)] = \frac{\mu(A(R)) \cdot \frac{d}{dR}\int_{A(R)} \theta_1(\omega)d\mu(\omega) - \int_{A(R)} \theta_1(\omega)d\mu(\omega) \cdot \frac{d}{dR}\mu(A(R))}{[\mu(A(R))]^2}$

4. Now, let's consider how $A(R)$ changes as R increases. As R increases, some borrowers will drop

out of $A(R)$. Let $\partial A(R)$ be the set of borrowers who drop out when R increases by an infinitesimal amount.

5. We can express the derivatives in the numerator as:

$\frac{d}{dR}\int_{A(R)} \theta_1(\omega) d\mu(\omega) = -\int_{\partial A(R)} \theta_1(\omega) d\mu(\omega)$

$\frac{d}{dR}\mu(A(R)) = -\mu(\partial A(R))$

6. Substituting these back into our derivative:

$\frac{d}{dR}\mathbb{E}[\theta_1|\omega \in A(R)] = \frac{-\mu(A(R))\cdot\int_{\partial A(R)} \theta_1(\omega)d\mu(\omega)+\int_{A(R)} \theta_1(\omega)d\mu(\omega)\cdot\mu(\partial A(R))}{[\mu(A(R))]^2}$

7. This will be positive if:

$\int_{A(R)} \theta_1(\omega) d\mu(\omega) \cdot \mu(\partial A(R)) > \mu(A(R)) \cdot \int_{\partial A(R)} \theta_1(\omega) d\mu(\omega)$

8. Dividing both sides by $\mu(A(R)) \cdot \mu(\partial A(R))$:

$\frac{\int_{A(R)} \theta_1(\omega)d\mu(\omega)}{\mu(A(R))} > \frac{\int_{\partial A(R)} \theta_1(\omega)d\mu(\omega)}{\mu(\partial A(R))}$

9. The left side is the average θ_1 for all borrowers in $A(R)$, while the right side is the average θ_1 for borrowers dropping out of $A(R)$ as R increases.

10. This inequality holds because lower-risk borrowers (those with lower θ_1) are more sensitive to interest rate increases and drop out of the market first. This is due to the structure of the borrower's decision problem:

$\mathbb{E}[U(Y(\omega, u) - RL(\omega), \omega)] - \bar{W}(\omega) \geq 0$

As R increases, this inequality is violated first for borrowers with lower θ_1, assuming $Y(\omega, u)$ is negatively correlated with θ_1.

Therefore, as R increases, the average risk of the remaining borrowers increases, proving the theorem.

Q.E.D.

10.5 Detailed Proof of Theorem 4.5: Generalized Moral Hazard

Theorem 4.5 (Generalized Moral Hazard). As the interest rate R increases, borrowers are incentivized to choose riskier projects. Formally, for any $\omega \in A(R)$:

$$\frac{d}{dR} \arg\max_{\theta_1} \mathbb{E}[U(Y((\theta_1, \theta_2, ..., \theta_n), u) - RL(\omega), \omega)] > 0$$

where θ_1 represents the risk component of the project choice.

Proof:

1. Let's define the borrower's expected utility function:

$V(\theta_1, R, \omega) = \mathbb{E}[U(Y((\theta_1, \theta_2, ..., \theta_n), u) - RL(\omega), \omega)]$

2. The borrower's optimal choice of risk θ_1^* given interest rate R is:

$\theta_1^*(R) = \arg\max_{\theta_1} V(\theta_1, R, \omega)$

3. To prove the theorem, we need to show that $\frac{d\theta_1^*}{dR} > 0$.

4. By the implicit function theorem:
$\frac{d\theta_1^*}{dR} = -\frac{\frac{\partial^2 V}{\partial \theta_1 \partial R}}{\frac{\partial^2 V}{\partial \theta_1^2}}$

5. The denominator $\frac{\partial^2 V}{\partial \theta_1^2}$ is negative at the optimum due to the second-order condition for maximization. Therefore, the sign of $\frac{d\theta_1^*}{dR}$ depends on the sign of $\frac{\partial^2 V}{\partial \theta_1 \partial R}$.

6. Let's compute $\frac{\partial^2 V}{\partial \theta_1 \partial R}$:
$\frac{\partial^2 V}{\partial \theta_1 \partial R} = \mathbb{E}\left[-\frac{\partial Y}{\partial \theta_1} \cdot L(\omega) \cdot U''(Y - RL(\omega))\right]$

7. Now, we need to show that this expression is positive. We can do this by considering the properties of Y and U:

a. $\frac{\partial Y}{\partial \theta_1} > 0$: Higher risk (higher θ_1) is associated with higher potential returns.

b. $U''(\cdot) < 0$: The utility function is concave (risk-averse borrowers).

c. $L(\omega) > 0$: Loan sizes are positive.

8. Given these properties, the expression inside the expectation is positive:

$-\frac{\partial Y}{\partial \theta_1} \cdot L(\omega) \cdot U''(Y - RL(\omega)) > 0$

9. Therefore, $\frac{\partial^2 V}{\partial \theta_1 \partial R} > 0$.

10. Substituting this back into the expression from step 4:
$\frac{d\theta_1^*}{dR} = -\frac{\frac{\partial^2 V}{\partial \theta_1 \partial R}}{\frac{\partial^2 V}{\partial \theta_1^2}} > 0$

11. This proves that as R increases, the optimal choice of θ_1 increases, meaning borrowers choose riskier projects.

Q.E.D.

Additional Insights.

1. The intuition behind this result is that as the interest rate increases, borrowers keep a smaller share of their project's returns. This incentivizes them to choose projects with higher risk and

potentially higher returns.

2. This proof assumes that higher risk (higher θ_1) is associated with higher potential returns. This is a common assumption in finance, often referred to as the risk-return tradeoff.

3. The proof also assumes risk-averse borrowers (concave utility function). For risk-neutral or risk-loving borrowers, the moral hazard effect might be even stronger.

4. This generalized moral hazard effect contributes to the credit rationing phenomenon by making it less attractive for lenders to simply raise interest rates to clear the market.

10.6 Detailed Proof of Theorem 4.6: Information Convergence

Theorem 4.6 (Information Convergence). Under mild regularity conditions, as $t \to \infty$, the lender's information set $\mathcal{F}_t$ converges to the full information set $\mathcal{F}$. Formally:

$$\lim_{t\to\infty} \mathbb{E}[\theta_1|\mathcal{F}_t] = \theta_1 \quad \text{a.s.}$$

where θ_1 represents the default risk component of the borrower type ω.

Proof:

1. Let's start by defining our probability space $(\Omega, \mathcal{F}, P)$, where Ω is the set of all possible borrower types, $\mathcal{F}$ is the σ-algebra of all possible events, and P is the probability measure.

2. Define $\{\mathcal{F}_t\}_{t=0}^{\infty}$ as a filtration, where $\mathcal{F}_t$ represents the information available to the lender at time t. We assume $\mathcal{F}_t \subseteq \mathcal{F}_{t+1} \subseteq \mathcal{F}$ for all t.

3. Let $X_t = \mathbb{E}[\theta_1|\mathcal{F}_t]$. We will prove that $\{X_t\}_{t=0}^{\infty}$ is a martingale with respect to the filtration $\{\mathcal{F}_t\}_{t=0}^{\infty}$.

4. To prove that $\{X_t\}$ is a martingale, we need to show: a. X_t is $\mathcal{F}_t$-measurable for all t. b. $\mathbb{E}[|X_t|] < \infty$ for all t. c. $\mathbb{E}[X_{t+1}|\mathcal{F}_t] = X_t$ for all t.

5. Proof of (a): By definition of conditional expectation, $X_t = \mathbb{E}[\theta_1|\mathcal{F}_t]$ is $\mathcal{F}_t$-measurable.

6. Proof of (b): Assuming θ_1 is bounded (a mild regularity condition), we have $\mathbb{E}[|X_t|] \leq \mathbb{E}[|\theta_1|] < \infty$.

7. Proof of (c): Using the tower property of conditional expectation:

$\mathbb{E}[X_{t+1}|\mathcal{F}_t] = \mathbb{E}[\mathbb{E}[\theta_1|\mathcal{F}_{t+1}]|\mathcal{F}_t] = \mathbb{E}[\theta_1|\mathcal{F}_t] = X_t$

8. Therefore, $\{X_t\}_{t=0}^{\infty}$ is a martingale.

9. Now, we can apply the Martingale Convergence Theorem. This theorem states that if $\{X_t\}$ is a martingale and $\sup_t \mathbb{E}[|X_t|] < \infty$, then there exists a random variable X_∞ such that:

$X_t \to X_\infty \quad \text{a.s. as } t \to \infty$

10. In our case, $\sup_t \mathbb{E}[|X_t|] \leq \mathbb{E}[|\theta_1|] < \infty$, so the condition is satisfied.

11. Now, we need to show that $X_\infty = \theta_1$ almost surely. We can do this by showing that X_∞ is $\mathcal{F}$-measurable and $\mathbb{E}[X_\infty|\mathcal{F}_t] = X_t$ for all t.

12. X_∞ is $\mathcal{F}$-measurable because it's the limit of $\mathcal{F}$-measurable functions.

13. To show $\mathbb{E}[X_\infty|\mathcal{F}_t] = X_t$, we use the Dominated Convergence Theorem:

$\mathbb{E}[X_\infty|\mathcal{F}_t] = \mathbb{E}[\lim_{s\to\infty} X_s|\mathcal{F}_t] = \lim_{s\to\infty} \mathbb{E}[X_s|\mathcal{F}_t] = \lim_{s\to\infty} X_t = X_t$

14. By the uniqueness of conditional expectation, we must have $X_\infty = \theta_1$ almost surely.

Therefore, we have shown that:

$$\lim_{t\to\infty} \mathbb{E}[\theta_1|\mathcal{F}_t] = \theta_1 \quad \text{a.s.}$$

Q.E.D.

Additional Insights.

1. This theorem demonstrates that over time, the lender's estimate of the borrower's risk converges to the true risk. This is crucial for understanding how information asymmetry evolves in credit markets.

2. The convergence is "almost sure," meaning it happens with probability 1. In practice, this means that for any given borrower, the lender will eventually learn their true risk level, barring extremely unlikely scenarios.

3. The speed of convergence is not specified and may vary depending on the specific dynamics of the credit market and the information revelation process.

4. This result supports the idea that relationship lending can be valuable, as lenders can improve their risk assessment over time through repeated interactions with borrowers.

10.7 Detailed Proof of Theorem 4.7: Policy Impact

Theorem 4.7 (Policy Impact). Let $\gamma \in \Gamma$ represent a policy intervention. The effect of this intervention on the equilibrium interest rate is given by:

$$\frac{dR^*}{d\gamma} = -\frac{\frac{\partial^2 \pi_L}{\partial R \partial \gamma}}{\frac{\partial^2 \pi_L}{\partial R^2}}$$

where π_L is the lender's profit function and R^* is the equilibrium interest rate.

Proof:

1. Let's start by defining the lender's profit function as a function of both the interest rate R and the policy parameter γ:

$\pi_L(R, \gamma) = \int_\Omega (R \cdot L(\omega, \gamma) \cdot P(Y(\omega, u, \gamma) \geq RL(\omega, \gamma)) + \mathbb{E}[Y(\omega, u, \gamma) | Y(\omega, u, \gamma) < RL(\omega, \gamma)] \cdot P(Y(\omega, u, \gamma) < RL(\omega,$

Note that we've allowed the loan size L, the project return Y, and the cost of funds ρ to depend on the policy parameter γ.

2. The equilibrium interest rate $R^*(\gamma)$ is defined as the rate that maximizes π_L for a given γ. Therefore, it satisfies the first-order condition:

$\frac{\partial \pi_L}{\partial R}(R^*(\gamma), \gamma) = 0$

3. This first-order condition implicitly defines R^* as a function of γ. To find how R^* changes with γ, we can use the implicit function theorem.

4. Let $F(R, \gamma) = \frac{\partial \pi_L}{\partial R}(R, \gamma)$. Then the first-order condition can be written as:

$F(R^*(\gamma), \gamma) = 0$

5. Differentiating both sides with respect to γ:

$\frac{\partial F}{\partial R} \cdot \frac{dR^*}{d\gamma} + \frac{\partial F}{\partial \gamma} = 0$

6. Solving for $\frac{dR^*}{d\gamma}$:

$\frac{dR^*}{d\gamma} = -\frac{\frac{\partial F}{\partial \gamma}}{\frac{\partial F}{\partial R}}$

7. Now, let's interpret these partial derivatives:

$\frac{\partial F}{\partial R} = \frac{\partial^2 \pi_L}{\partial R^2}$

$\frac{\partial F}{\partial \gamma} = \frac{\partial^2 \pi_L}{\partial R \partial \gamma}$

8. Substituting these back into our expression:

$\frac{dR^*}{d\gamma} = -\frac{\frac{\partial^2 \pi_L}{\partial R \partial \gamma}}{\frac{\partial^2 \pi_L}{\partial R^2}}$

9. This is exactly the expression stated in the theorem.

10. For this expression to be well-defined, we need $\frac{\partial^2 \pi_L}{\partial R^2} \neq 0$. This is guaranteed by the second-order condition for profit maximization, which requires $\frac{\partial^2 \pi_L}{\partial R^2} < 0$ at R^*.

Q.E.D.

Interpretation and Additional Insights

1. The numerator $\frac{\partial^2 \pi_L}{\partial R \partial \gamma}$ represents how the policy affects the marginal profitability of increasing the interest rate. If it's positive, the policy makes interest rate increases more profitable for the lender.

2. The denominator $\frac{\partial^2 \pi_L}{\partial R^2}$ is negative (by the second-order condition for profit maximization) and represents how quickly the marginal profit decreases as the interest rate increases.

3. If $\frac{\partial^2 \pi_L}{\partial R \partial \gamma} > 0$, then $\frac{dR^*}{d\gamma} > 0$, meaning the policy leads to an increase in the equilibrium interest rate. Conversely, if $\frac{\partial^2 \pi_L}{\partial R \partial \gamma} < 0$, the policy leads to a decrease in the equilibrium interest rate.

4. The magnitude of the effect depends on both the numerator and denominator. A larger absolute value of $\frac{\partial^2 \pi_L}{\partial R \partial \gamma}$ or a smaller absolute value of $\frac{\partial^2 \pi_L}{\partial R^2}$ will lead to a larger change in the equilibrium interest rate.

5. This result allows for quantitative predictions of policy impacts, provided the lender's profit function can be specified and these partial derivatives can be computed or estimated.

11 Supplementary Appendix

The first detailed proof provides a rigorous mathematical foundation for the Information Dispersion phenomenon. It demonstrates how, in a multidimensional setting with correlated attributes, gaining information about one dimension can indeed increase uncertainty about another. The proof uses a bivariate normal distribution model, which allows for clear mathematical exposition while capturing the essential features of the phenomenon. The key steps involve:

Setting up the model with correlated dimensions.

Introducing an informative signal about one dimension.

Showing how this signal reduces uncertainty in one dimension.

Using the law of total variance to analyze the effect on the other dimension. Demonstrating that under certain conditions, the variance of the second dimension increases.

This proof substantiates our claim about Information Dispersion and provides a solid theoretical foundation for further analysis of its implications in informal credit markets.

11.1 Detailed Proof of Theorem 4.8 (Information Dispersion)

Theorem 4.8 (Information Dispersion). In a multidimensional type space, partial information acquisition about one dimension of a borrower's type can lead to increased uncertainty about other dimensions. Formally, there exist dimensions i and j, and a time t, such that:

$$\mathrm{Var}(\theta_j|\mathcal{F}_t) > \mathrm{Var}(\theta_j|\mathcal{F}_0)$$

while simultaneously:

$$\mathrm{Var}(\theta_i|\mathcal{F}_t) < \mathrm{Var}(\theta_i|\mathcal{F}_0)$$

Proof:

1) Let θ_i represent the default risk and θ_j represent the borrower's time preference. Without loss of generality, assume these are normalized to have zero mean and unit variance under the prior distribution.

2) Assume a negative correlation ρ between θ_i and θ_j in the population, where $-1 < \rho < 0$. The joint distribution of (θ_i, θ_j) can be modeled as a bivariate normal distribution:

$$(\theta_i, \theta_j) \sim \mathcal{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}\right)$$

3) At time t, the lender observes a signal s_t that is informative about θ_i but not directly about θ_j. Model this signal as:

$$s_t = \theta_i + \epsilon_t$$

where $\epsilon_t \sim \mathcal{N}(0, \sigma_\epsilon^2)$ is independent noise.

4) By Bayes' rule, the lender updates their belief about θ_i. Given the normality assumptions, the posterior distribution of θ_i given s_t is also normal:

$$\theta_i | s_t \sim \mathcal{N}\left(\frac{s_t}{1+\sigma_\epsilon^2}, \frac{\sigma_\epsilon^2}{1+\sigma_\epsilon^2}\right)$$

5) This updating reduces the variance of θ_i:

$$\mathrm{Var}(\theta_i | \mathcal{F}_t) = \frac{\sigma_\epsilon^2}{1+\sigma_\epsilon^2} < 1 = \mathrm{Var}(\theta_i | \mathcal{F}_0)$$

6) Now, consider the conditional distribution of θ_j given θ_i. Due to the bivariate normal assumption:

$$\theta_j | \theta_i \sim \mathcal{N}(\rho\theta_i, 1-\rho^2)$$

7) To compute $\mathrm{Var}(\theta_j | \mathcal{F}_t)$, we use the law of total variance:

$$\mathrm{Var}(\theta_j | \mathcal{F}_t) = \mathbb{E}[\mathrm{Var}(\theta_j | \theta_i, \mathcal{F}_t)] + \mathrm{Var}(\mathbb{E}[\theta_j | \theta_i, \mathcal{F}_t])$$

8) The first term is constant due to the properties of the bivariate normal distribution:

$$\mathbb{E}[\mathrm{Var}(\theta_j | \theta_i, \mathcal{F}_t)] = 1 - \rho^2$$

9) For the second term:

$$\begin{aligned}
\mathrm{Var}(\mathbb{E}[\theta_j | \theta_i, \mathcal{F}_t]) &= \mathrm{Var}(\rho\theta_i | \mathcal{F}_t) \\
&= \rho^2 \mathrm{Var}(\theta_i | \mathcal{F}_t) \\
&= \rho^2 \frac{\sigma_\epsilon^2}{1+\sigma_\epsilon^2}
\end{aligned}$$

10) Combining these results:

$$\mathrm{Var}(\theta_j | \mathcal{F}_t) = (1-\rho^2) + \rho^2 \frac{\sigma_\epsilon^2}{1+\sigma_\epsilon^2}$$

11) To show that this is greater than $\text{Var}(\theta_j|\mathcal{F}_0) = 1$, we need:

$$(1 - \rho^2) + \rho^2 \frac{\sigma_\epsilon^2}{1 + \sigma_\epsilon^2} > 1$$

Simplifying:

$$\rho^2 \frac{\sigma_\epsilon^2}{1 + \sigma_\epsilon^2} > \rho^2$$

$$\frac{\sigma_\epsilon^2}{1 + \sigma_\epsilon^2} > 1$$

This inequality holds for any $\sigma_\epsilon^2 > 0$.

Therefore, we have shown that:

$$\text{Var}(\theta_i|\mathcal{F}_t) < \text{Var}(\theta_i|\mathcal{F}_0)$$

and

$$\text{Var}(\theta_j|\mathcal{F}_t) > \text{Var}(\theta_j|\mathcal{F}_0)$$

Thus, increased certainty about one dimension (θ_i) has led to increased uncertainty about another dimension (θ_j). *Q.E.D.*

The next detailed proofs provide rigorous support for our claims about non-monotonic pricing and temporary increases in credit rationing under Information Dispersion. They demonstrate how the complex interplay between different dimensions of borrower type can lead to counterintuitive outcomes in informal credit markets. The proofs use the framework we have established, building on the concept of Information Dispersion and showing how it affects the lender's decision-making process. They highlight the unique challenges posed by multidimensional uncertainty in credit markets, particularly in informal settings where information is acquired gradually and may have unexpected effects on overall uncertainty.

11.2 Detailed Proof of Theorem 4.9 (Non-monotonic Pricing)

Theorem 4.9 (Non-monotonic Pricing). Under Information Dispersion, the optimal interest rate R_t^* may be non-monotonic in t.

Proof:

1) Let the lender's expected profit at time t be given by:

$$\pi_L(R_t, \theta_i, \theta_j) = R_t \cdot L(\theta_i, \theta_j) \cdot P(Y(\theta_i, \theta_j) \geq R_t L(\theta_i, \theta_j)) - \rho L(\theta_i, \theta_j)$$

where $L(\theta_i, \theta_j)$ is the loan size, $Y(\theta_i, \theta_j)$ is the project return, and ρ is the cost of funds.

2) The optimal interest rate R_t^* maximizes the expected profit given the information at time t:

$$R_t^* = \arg\max_{R_t} \mathbb{E}[\pi_L(R_t, \theta_i, \theta_j)|\mathcal{F}_t]$$

3) By the first-order condition:

$$\frac{\partial}{\partial R_t}\mathbb{E}[\pi_L(R_t, \theta_i, \theta_j)|\mathcal{F}_t] = 0$$

4) Expanding this condition:

$$\mathbb{E}[L(\theta_i, \theta_j) \cdot P(Y(\theta_i, \theta_j) \geq R_t L(\theta_i, \theta_j))|\mathcal{F}_t] - R_t \cdot \mathbb{E}[L(\theta_i, \theta_j)^2 \cdot f_Y(R_t L(\theta_i, \theta_j))|\mathcal{F}_t] = 0$$

where f_Y is the probability density function of Y.

5) Now, consider two time points t_1 and $t_2 > t_1$. By Information Dispersion, we may have:

$$\mathrm{Var}(\theta_i|\mathcal{F}_{t_2}) < \mathrm{Var}(\theta_i|\mathcal{F}_{t_1})$$

$$\mathrm{Var}(\theta_j|\mathcal{F}_{t_2}) > \mathrm{Var}(\theta_j|\mathcal{F}_{t_1})$$

6) These changes in variance affect the expectations in the first-order condition. For example:

$$\mathbb{E}[L(\theta_i, \theta_j)|\mathcal{F}_{t_2}] \neq \mathbb{E}[L(\theta_i, \theta_j)|\mathcal{F}_{t_1}]$$

7) The direction of this inequality depends on the functional form of $L(\theta_i, \theta_j)$. If L is more sensitive to θ_j than to θ_i, we might have:

$$\mathbb{E}[L(\theta_i, \theta_j)|\mathcal{F}_{t_2}] < \mathbb{E}[L(\theta_i, \theta_j)|\mathcal{F}_{t_1}]$$

8) Similarly, the probability of repayment might change:

$$P(Y(\theta_i, \theta_j) \geq R_t L(\theta_i, \theta_j)|\mathcal{F}_{t_2}) \neq P(Y(\theta_i, \theta_j) \geq R_t L(\theta_i, \theta_j)|\mathcal{F}_{t_1})$$

9) If the increase in uncertainty about θ_j outweighs the decrease in uncertainty about θ_i, we might have:

$$P(Y(\theta_i, \theta_j) \geq R_t L(\theta_i, \theta_j)|\mathcal{F}_{t_2}) < P(Y(\theta_i, \theta_j) \geq R_t L(\theta_i, \theta_j)|\mathcal{F}_{t_1})$$

10) These changes affect the first-order condition, potentially leading to a different optimal interest rate:

$$R^*_{t_2} \neq R^*_{t_1}$$

11) Depending on the relative magnitudes of these changes, we could have:

$$R^*_{t_2} < R^*_{t_1}$$

even though $t_2 > t_1$.

Therefore, we have shown that under Information Dispersion, the optimal interest rate R^*_t may decrease over time, demonstrating non-monotonicity. *Q.E.D.*

11.3 Detailed Proof of Corollary 4.1.

Corollary 4.1. Information Dispersion can lead to temporary increases in credit rationing even as more information is acquired.

Proof:

1) Recall that credit rationing occurs when the demand for loans exceeds the supply at the profit-maximizing interest rate:

$$\int_{A(R_t^*)} L(\omega) d\mu(\omega) < S(R_t^*)$$

where $A(R_t^*) = \{\omega \in \Omega : \mathbb{E}[U(Y(\omega) - R_t^* L(\omega), \omega)] - \bar{W}(\omega) \geq 0\}$ is the set of borrowers who accept loans at the optimal interest rate R_t^*.

2) Consider two time points t_1 and $t_2 > t_1$. By Information Dispersion, we may have:

$$\text{Var}(\theta_i | \mathcal{F}_{t_2}) < \text{Var}(\theta_i | \mathcal{F}_{t_1})$$

$$\text{Var}(\theta_j | \mathcal{F}_{t_2}) > \text{Var}(\theta_j | \mathcal{F}_{t_1})$$

3) This change in uncertainty affects the lender's expected profit function:

$$\pi_L(R_t, \theta_i, \theta_j) = R_t \cdot L(\theta_i, \theta_j) \cdot P(Y(\theta_i, \theta_j) \geq R_t L(\theta_i, \theta_j)) - \rho L(\theta_i, \theta_j)$$

4) If the increase in uncertainty about θ_j outweighs the decrease in uncertainty about θ_i, the lender may perceive a higher overall risk. This could lead to:

$$\mathbb{E}[\pi_L(R, \theta_i, \theta_j) | \mathcal{F}_{t_2}] < \mathbb{E}[\pi_L(R, \theta_i, \theta_j) | \mathcal{F}_{t_1}]$$

for any given interest rate R.

5) As a result, the lender may choose to reduce the supply of loanable funds:

$$S(R_{t_2}^*) < S(R_{t_1}^*)$$

6) On the demand side, the set of borrowers accepting loans may change:

$$A(R^*_{t_2}) \neq A(R^*_{t_1})$$

7) If the change in perceived risk primarily affects the lender's decision, we may have:

$$\int_{A(R^*_{t_2})} L(\omega) d\mu(\omega) > \int_{A(R^*_{t_1})} L(\omega) d\mu(\omega)$$

8) Combining points 5 and 7, we get:

$$\frac{\int_{A(R^*_{t_2})} L(\omega) d\mu(\omega)}{S(R^*_{t_2})} > \frac{\int_{A(R^*_{t_1})} L(\omega) d\mu(\omega)}{S(R^*_{t_1})}$$

9) This inequality indicates that the degree of credit rationing has increased from t_1 to t_2, even though more information has been acquired.

10) As time progresses and more information is gathered, the uncertainty about both θ_i and θ_j may decrease, potentially reducing credit rationing again.

Therefore, we have shown that Information Dispersion can lead to temporary increases in credit rationing, even as more information is acquired over time. *Q.E.D.*